CNCM Generating Functions Handout

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§1 What are Generating Functions?

Generating Functions are a powerful probability tool that can help us represent probability with the use of a polynomial. Let's take a look at a term of a generating function, and what it represents. Take the term cx^k.

- The exponent k represents a state, or an event that can occur
- The coefficient c represents the probability that the state represented by k can occur

You can find out the number of ways that an event can occur by factoring out the probability. This is best shown with an example.

§2 Generating Functions with Dice

Let's take a look at dice rolls using generating functions. The possible states that can occur on a standard dice roll are $1, 2, 3, 4, 5, 6$. These all occur with equal probability $\frac{1}{6}$. Thus, we can represent the dice roll with the generating function:

$$\frac{1}{6}x + \frac{1}{6}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{6}x^5 + \frac{1}{6}x^6$$

As you can see, the exponents represent the possible states, and the coefficients represent the probability that each state occurs.

> **Example 2.1** — What is the probability of rolling a sum of 5 on two standard dice?

This is where the power of generating functions comes in. We can represent the sum of rolling two dice with a generating function as well. We simply doing this by multiplying the generating functions for the roll of a standard dice, or:

$$\left(\frac{1}{6}x + \frac{1}{6}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{6}x^5 + \frac{1}{6}x^6\right)^2$$

Here, we can factor out $\frac{1}{36}$ to get the number of ways that each sum occurs instead of the probability:

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^2$$

Why does this work? Let's think about how we can achieve a sum of 5. We have four combinations, 4 on the first dice and 1 on the second dice, 3 on the first dice and 2 on the second dice, 2 on the first dice and 3 on the second dice, and 1 on the first dice and 4 on the second dice.

The generating function works similarly. When we expand this, the following combinations of exponents give x^5: $x^4 \cdot x, x^3 \cdot x^2, x^2 \cdot x^3, x^4 \cdot x$. There are 4 such combinations, so the coefficient of x^5 will be 4.

As you can see, the possible events that lead to a sum of 5 are encoded in terms of a polynomial. Thus, all we had to do in this problem is find the coefficient of x^5 (which corresponds to a state of the sum of 5) in the expansion. We got that the coefficient was 4, and thus the total probability is $\frac{4}{36} = \boxed{\frac{1}{9}}$.

Obviously, this seems like a tedious way to do a simple problem, but becomes much more useful in problems with higher numbers. In this case, we don't have a nice way of figuring out coefficients as the Binomial Theorem doesn't work here, but future problems will have nice ways of doing so.

Problem 2.2

(2010 AIME I 4) Jackie and Phil have two fair coins and a third coin that comes up heads with probability $\frac{4}{7}$. Jackie flips the three coins, and then Phil flips the three coins. Let $\frac{m}{n}$ be the probability that Jackie gets the same number of heads as Phil, where m and n are relatively prime positive integers. Find $m + n$.

Here, casework seems pretty tedious. Let's use generating functions. Here, we would like to know the probability of getting 0, 1, 2, or 3 heads when flipping three coins. We can represent this with a generating function, with each state being the number of heads flipped.

Let's look at the setup for a fair coin. We flip tails with probability $\frac{1}{2}$ and heads with probability $\frac{1}{2}$. Thus, there is a $\frac{1}{2}$ chance of a fair coin contributing 1 towards our count of heads. We have

$$\frac{1}{2} + \frac{1}{2}x$$

Similarly, we get the following generating function for the unfair coin using the probabilities given in the problem:

$$\frac{3}{7} + \frac{4}{7}x$$

. Thus, we can represent the number of heads achieved by the flips of the three coins by multiplying their individual generating functions together to get

$$\left(\frac{1}{2} + \frac{1}{2}x\right)^2 \left(\frac{3}{7} + \frac{4}{7}x\right)$$

Expanding this out, we get

$$\frac{4}{28}x^3 + \frac{11}{28}x^2 + \frac{10}{28}x + \frac{3}{28}$$

We now know the probabilities of flipping $0, 1, 2,$ or 3 heads from flipping these coins. To finish the problem, we just square each of these probabilities as we need *both* Jackie and Phil to get k number of heads. Our answer ends up being

$$\frac{4^2 + 11^2 + 10^2 + 3^2}{28^2} = \frac{16 + 121 + 100 + 9}{784} = \frac{246}{784} = \frac{123}{392}$$

So our answer is just $123 + 392 = \boxed{515}$.

§3 Power Series

The generating functions that we have been working with so far have been finite, as there were a limited amount of states. What if we wanted to represent a quantity with an *infinite* amount of states? We could do so with an infinite polynomial, such as:

$$1 + x + x^2 + x^3 + x^4 \dots$$

This generating function, in fact can represent an integer in the context of summing integers. Say we wanted to find the number of nonnegative solutions (a, b, c, d) of $a + b + c + d = 3$. We can represent each of a, b, c, d with the above polynomial, as each exponent corresponds to an integer.

Based on what we've learned about generating functions, our answer would just be the coefficient of x^3 in the expansion of

$$(1 + x + x^2 + x^3 + x^4 \dots)^4$$

We could just calculate this by hand, as there are not many combinations that multiply to x^3. However, what if we wanted to deal with numbers larger than 3? This is where the power series comes into play.

Remark 3.1

$$1 + x + x^2 + x^3 + x^4 \dots = \frac{1}{1-x}$$

The proof of this is very simple. Given $|x| < 1$, the series diverges and we can use the sum of an infinite geometric series formula with initial term 1 and common ratio x.

This means we can represent our problem from earlier with the polynomial

$$\left(\frac{1}{1-x}\right)^4$$

There is indeed a nice way to find the coefficients of this expansion, and it comes from the **Negative Binomial Theorem**.

Remark 3.2

$$\left(\frac{1}{1-x}\right)^n = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

Revisiting our problem of the number of nonnegative solutions (a, b, c, d) of $a + b + c + d = 10$, we get that our answer is the coefficient of x^10 in the expansion

$$\left(\frac{1}{1-x}\right)^4$$

Which is just

$$\binom{4+10-1}{10} = \binom{13}{10}$$

Problem 3.3

Find the number of *positive* integer solutions (a, b, c, d) to $a + b + c + d = 10$.

This is very similar to our earlier problem, just with the constraint positive. This means we no longer have the x^0 state in our generating function. Thus, our answer is just the coefficient of x^10 in the expansion

$$(x + x^2 + x^3 + \dots)^4 = x^4(1 + x + x^2 + \dots) = \frac{x^4}{(1-x)^4}$$

Which can be rewritten as finding the coefficient of x^6 in the expansion of

$$\left(\frac{1}{1-x}\right)^4$$

Which we know is just

$$\binom{4+6-1}{6} = \boxed{\binom{9}{6}}$$

§4 Roots of Unity Filter

The Roots of Unity filter is a very powerful tool when combined with the use of generating functions. The filter gives us the sum of the coefficients of every x^n such that n is divisible by k. This sum can be written as

Remark 4.1

$$\frac{P(1) + P(\omega) + P(\omega^2) + \dots + P(\omega^{k-1})}{k}$$

Where $P(x)$ is a polynomial and $1, \omega, \omega^2, \dots, \omega^{k-1}$ are the kth roots of unity.

The above computation may seem a bit ugly, but using the basic facts about Roots of Unity that $1 + \omega + \omega^2 + \dots + \omega^{k-1} = 0$ for any root of unity other than 1, and that $w^k = 1$, the very basic definition of a root of unity. Let's cover a basic example to show this in action.

Example 4.2 — How many ways are there to roll three standard dice such that the sum of the numbers that are rolled is divisible by 3?

We know the generating function for a standard dice, so our answer is just the sum of the coefficients of all x^k such that k is divisible by 3 in the expansion

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^3$$

This is where the Roots of Unity filter comes into play. We know that our generating function is $P(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^3$, and we want to use the filter for $k = 3$. We just need to compute the following

$$\frac{P(1) + P(\omega) + P(\omega^2)}{3}$$

We have the following:

$$P(1) = 6^3 = 216$$

$$P(\omega) = (\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6)^3 = (\omega + \omega^2 + 1 + \omega + \omega^2 + 1)^3 = (0 + 0)^3 = 0$$

$$P(\omega^2) = (\omega^2 + \omega^4 + \omega^6 + \omega^8 + \omega^10 + \omega^12)^3 = (\omega + \omega^2 + 1 + \omega + \omega^2 + 1)^3 = (0 + 0)^3 = 0$$

Thus, our answer is just

$$\frac{216 + 0 + 0}{3} = \boxed{72}$$

Problem 4.3

(2018 AIME I 12) For every subset T of $U = 1, 2, 3, \ldots, 18$, let $s(T)$ be the sum of the elements of T, with $s(\emptyset)$ defined to be 0. If T is chosen at random among all subsets of U, the probability that $s(T)$ is divisible by 3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

Let's try to find a generating function for the sum of the elements of a subset T. We know that we can form subsets using the "in or out" argument, where we choose if an element from U is either in or out of our subset T. We can try the same idea here. Let's look at the element 1. If 1 is out of our subset, then it contributes 0 to the sum, so we will just represent it with a 1. If it is in the subset, then it will contribute 1 to the sum, so we can write this as x^1.

The same argument can be made for every other element, so our generating function is

$$P(x) = (1 + x)(1 + x^2)(1 + x^3) \ldots (1 + x^{18})$$

We are just counting the number of ways for now, so we don't have any probabilities attached.

Now we need to sum to be divisible by 3, which is equivalent to finding the sum of the coefficients of all x^k in the expansion such that k is divisible by 3. We use the filter once again, so we need to compute

$$\frac{P(1) + P(\omega) + P(\omega^2)}{3}$$

We have:

$$P(1) = 2^{18}$$

$$P(\omega) = (1 + \omega)(1 + \omega^2)(1 + \omega^3) \ldots (1 + \omega^{18}) = ((1 + \omega)(1 + \omega^2)(1 + 1))^6 = (2(w^3 + w^2 + w + 1))^6 = 2^6$$

$$P(\omega^2) = (1 + \omega^2)(1 + \omega^4)(1 + \omega^6) \ldots (1 + \omega^{36}) = ((1 + \omega)(1 + \omega^2)(1 + 1))^6 = (2(w^3 + w^2 + w + 1))^6 = 2^6$$

So the number of subsets that have a sum divisible by 3 are:

$$\frac{2^{18} + 2^6 + 2^6}{3}$$

. Since we are trying to calculate the probability, we divide by the total number of possible subsets, which is 2^{18}, to get:

$$\frac{\frac{2^{18} + 2^6 + 2^6}{3}}{2^{18}} = \frac{\frac{2^{12} + 1 + 1}{3}}{2^{12}} = \frac{4098}{3 \cdot 4096} = \frac{1366}{4096} = \frac{683}{2048}$$

So our answer is just $\boxed{683}$.

§5 Problems of the Day

Problem 5.1. (2019 CNCM Math Bowl Indiv 9) Three regular 7-sided dice, two regular 5-sided dice, and one regular 4-sided die are rolled. The probability that the 6 dice sum to a number divisible by 3 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

Problem 5.2. The value

$$\sum_{j=4}^{\infty} \frac{\binom{j}{4}}{4^j}$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 5.3. (2007 HMMT Combo 9) Let S denote the set of all triples (a, b, c) of positive integers where $a+b+c = 15$. Compute

$$\sum_{(a,b,c)\in S} abc$$

Problem 5.4. (2013 HMMT Algebra 7) The value

$$\sum_{a_1=0}^{\infty} \sum_{a_2=0}^{\infty} \cdots \sum_{a_7=0}^{\infty} \frac{a_1+a_2+\cdots+a_7}{3^{a_1+a_2+\cdots+a_7}}$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 5.5. (2008 HMMT Combo 9) Determine the number of 8-tuples of nonnegative integers $(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ satisfying $0 \le a_k \le k$, for each $k = 1, 2, 3, 4$, and $a_1 + a_2 + a_3 + a_4 + 2b_1 + 3b_2 + 4b_3 + 5b_4 = 19$

§6 Solutions to the Problems

PoTD 1

(2019 CNCM Math Bowl Indiv 9) Three regular 7-sided dice, two regular 5-sided dice, and one regular 4-sided die are rolled. The probability that the 6 dice sum to a number divisible by 3 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

This is very similar to the dice problem we covered in the handout. We know how to find the generating functions for each dice. Our generating function is:

$$P(x) = (x+x^2+x^3+x^4+x^5+x^6+x^7)^3(x+x^2+x^3+x^4+x^5)^2(x+x^2+x^3+x^4)$$

Since we are trying to find the number of ways such that the dice sum to 3, we will use the filter for $k = 3$. We have:

$$P(1) = 7^3 \cdot 5^2 \cdot 4$$

$$P(\omega) = (\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6+\omega^7)^3(\omega+\omega^2+\omega^3+\omega^4+\omega^5)^2(\omega+\omega^2+\omega^3+\omega^4) = (\omega)^3(\omega+\omega^2)^2(\omega) = \omega$$

$$P(\omega^2) = (\omega^2+\omega^4+\omega^6+\omega^8+\omega^{10}+\omega^{12}+\omega^{14})^3(\omega^2+\omega^4+\omega^6+\omega^8+\omega^{10})^2(\omega^2+\omega^4+\omega^6+\omega^8) = (\omega^2)^3(\omega+\omega^2)^2(\omega^2) = \omega^2$$

The number of ways is

$$\frac{P(1)+P(\omega)+P(\omega^2)}{3} = \frac{7^3 \cdot 5^2 \cdot 4 + \omega + \omega^2}{3} = \frac{7^3 \cdot 5^2 \cdot 4 - 1}{3}$$

by using the fact $\omega^2 + \omega + 1 = 0 \implies \omega^2 + \omega = -1$. The total number of ways to roll the dice is $7^3 \cdot 5^2 \cdot 4$, so we divide to find the probability:

$$\frac{\frac{7^3 \cdot 5^2 \cdot 4 - 1}{3}}{7^3 \cdot 5^2 \cdot 4} = \frac{11433}{7^3 \cdot 5^2 \cdot 4}$$

Since the problem just asks for m, our answer is $\boxed{11433}$.

PoTD 2

The value

$$\sum_{j=4}^{\infty} \frac{\binom{j}{4}}{4^j}$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

The problem convinces us to make the substitution $j = k + 4$. Let's also substitute in $x = \frac{1}{4}$ we now have the summation

$$\sum_{k=0}^{\infty} \binom{k+4}{4} x^{k+4}$$

This looks suspiciously like an expansion of $\left(\frac{1}{1-x}\right)^n$. We know that $\binom{k+4}{4} = \binom{k+4}{k} = \binom{k+5-1}{k}$.

This looks like the expansion of

$$\left(\frac{1}{1-x}\right)^5$$

We know that:

$$\left(\frac{1}{1-x}\right)^5 = \sum_{k=0}^{\infty} \binom{k+5-1}{k} x^k$$

But our summation is

$$\sum_{k=0}^{\infty} \binom{k+5-1}{k} x^{k+4}$$

For the summations to match, we need to multiply by x^4. Thus, the closed form for the expression is

$$\frac{x^4}{(1-x)^5}$$

Substituting in $x = \frac{1}{4}$, we have:

$$\frac{\left(\frac{1}{4}\right)^4}{\left(1-\frac{1}{4}\right)^5} = \frac{4}{243}$$

Thus, our answer is just $4 + 243 = \boxed{247}$.

PoTD 3

(2007 HMMT Combo 9) Let S denote the set of all triples (a, b, c) of positive integers where $a + b + c = 15$. Compute

$$\sum_{(a,b,c)\in S} abc$$

This problem seems very similar to our problem of finding the positive solutions (a, b, c, d) to $a+b+c+d = 10$. We can take a closer look at what happens with $a + b + c = 15$.

When we represent this problem with generating functions, we get that the number of ways is equivalent to the coefficient because the product that represents the solution $(a, b, c) = (3, 8, 4)$ is $x^3 \cdot x^8 \cdot x^4$ in our generating function. This is true for all such solutions, and when we add them all up, we get the number of ways since all of the coefficients are 1.

In our case, we want to sum the product of the exponents when we sum up all the possibilities. Since we know that coefficients get multiplied and exponents get added when we expand our generating function, we can do so with the following generating function:

$$x + 2x^2 + 3x^3 + 4x^4 + \dots$$

Let's see why this works. Going back to our $(a, b, c) = (3, 8, 4)$ example, we now multiply $3x^3 \cdot 8x^8 \cdot 4x^4 = 96x^{15}$ for this possibility. The coefficient ends up being the product of the triple, so if we sum up all possible triples, we get the sum of all possible products.

The problem boils down to finding the coefficient of x^{15} in the generating function

$$(x + 2x^2 + 3x^3 + 4x^4 + \dots)^3 = x^3(1 + 2x + 3x^2 + 4x^3 + \dots)^3$$

The closed form of $1 + 2x + 3x^2 + 4x^3 + \dots$ is $\frac{1}{(1-x)^2}$ (if you aren't convinced, try it yourself). Now, we just need to find the coefficient of x^{15} in

$$\frac{x^3}{(1-x)^6}$$

Which is the coefficient of x^{12} in

$$\frac{1}{(1-x)^6}$$

We know that this is just

$$\binom{6+12-1}{12} = \binom{17}{12} = \boxed{6188}$$

PoTD 4

(2013 HMMT Algebra 7) The value

$$\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_7=0}^{\infty}\frac{a_1+a_2+\cdots+a_7}{3^{a_1+a_2+\cdots+a_7}}$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Let's disregard the $a_1+a_2+\cdots+a_7$ for now and make the substitution $x=\frac{1}{3}$. Our summation is now

$$\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_7=0}^{\infty}x^{a_1+a_2+\cdots+a_7}$$

Let's think about this logically. Assume that $a_1+a_2+\cdots+a_7=k$. The summation adds $x^{a_1+a_2+\cdots+a_7}$ for every possible set of $(a_1,a_2,a_3,a_4,a_5,a_6,a_7)$. This is indeed the generating function

$$(1+x+x^2+x^3+\dots)^7=\frac{1}{(1-x)^7}$$

We need to account for the $a_1+a_2+\cdots+a_7$ we disregarded earlier. Let's convert our generating function to a summation

$$\frac{1}{(1-x)^7}=\sum_{k=0}^{\infty}\binom{k+6}{k}x^k$$

Since we know that $a_1+a_2+\cdots+a_7=k$, we can write it in the summation as

$$\sum_{k=0}^{\infty}k\binom{k+6}{k}x^k$$

. To get the k in the expression, we can differentiate and multiply by x:

$$F(x)=\sum_{k=0}^{\infty}\binom{k+6}{k}x^k$$

$$xF'(x)=\sum_{k=0}^{\infty}k\binom{k+6}{k}x^k$$

Thus, our answer is just substituting $\frac{1}{3}$ into $xF'(x)=\frac{7x}{(1-x)^8}$, which is just:

$$\frac{7\left(\frac{1}{3}\right)}{\left(1-\frac{1}{3}\right)^8}=\frac{15309}{256}$$

Thus our answer is just $15309+256=\boxed{15565}$.

PoTD 5

(2008 HMMT Combo 9) Determine the number of 8-tuples of nonnegative integers $(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ satisfying $0 \le a_k \le k$, for each $k = 1, 2, 3, 4$, and $a_1 + a_2 + a_3 + a_4 + 2b_1 + 3b_2 + 4b_3 + 5b_4 = 19$

We have that $0 \le a_1 \le 1, 0 \le a_2 \le 2, 0 \le a_3 \le 3, 0 \le a_4 \le 4$. Thus, our generating functions for the first four variables are

$$(1+x)(1+x+x^2)(1+x+x^2+x^3)(1+x+x^2+x^3+x^4)$$

Our last four variables are unconstrained, so we can write them as power series. Since the variables are multiples of $2, 3, 4, 5$ respectively, we only take multiples of those numbers into account, giving us the generating function:

$$(1+x^2+x^4+\dots)(1+x^3+x^6+\dots)(1+x^4+x^8+\dots)(1+x^5+x^{10}+\dots)$$

We can write $(1+x+x^2+\cdots+x^k)$ as $\dfrac{x^{k+1}-1}{x-1} = \dfrac{1-x^k}{1-x}$,so we get:

$$\left(\frac{1-x^2}{1-x}\right)\left(\frac{1-x^3}{1-x}\right)\left(\frac{1-x^4}{1-x}\right)\left(\frac{1-x^5}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^4}\right)\left(\frac{1}{1-x^5}\right)$$

Through convenient cancellation, this all simplifies down to $\dfrac{1}{(1-x)^4}$. We need to find the coefficient of x^{19} in this expansion, which we know is just:

$$\binom{4+19-1}{19} = \binom{22}{19} = \boxed{1540}$$